Extending constrained free energy to real space

- We know that, $E(J) = \sum_{\mu} \theta\left(-\frac{\sigma_0^{\mu}}{\sqrt{N}} \sum_{i=1}^{N} J_i \xi_i^{\mu}\right)$

since it's only a matter of sign (+/-)
if 'J' is a particular sol^n then so is 'cJ'... $(c > 0)$

$$\because \quad \theta\left(-\frac{c\sigma_0^{\mu}}{\sqrt{N}} \sum_{i=1}^{N} J_i \xi_i^{\mu}\right) = \theta\left(\frac{-\sigma_0}{\sqrt{N}} \sum_{i=1}^{N} J_i \xi_i^{\mu}\right)$$

- We thus try to constrain J/W to a certain sol^n volume, which acts as normalization.

or
we assume (pessimistically)

1). $\sum_j J_{ij}^2 < N \implies \theta\left(N - \sum_j J_{ij}^2\right)$

2). $\sum_j W_{ij}^2 < N \implies \theta\left(N - \sum_j W_{ij}^2\right)$

- Going forward we'll use following approximations:

1). $\ln Z = \lim_{n \to 0} \frac{\partial Z^n}{\partial n}$ / 2). $Z^{-1} = \lim_{n \to 0} Z^{n-1}$.

⊙ Let, a). $E^a = 1 - \sum_{i=1}^{N} \frac{(J_i^a)^2}{N}$ $\Rightarrow$ let $E_a = e$, $\hat{E}_a = \hat{e}$

b). $F^r = 1 - \sum_{i=1}^{N} \frac{(W_i^r)^2}{N}$ $\Rightarrow$ let $F_r = f$, $\hat{F}_r = \hat{f}$

⊙ The expression for free energy that we finally get is:

$$F(n) = \left\langle \left[\left(\prod_{a=1}^{n} \int dJ^a \prod_{r=1}^{m} \int dW^r \right) \prod_{\mu} \left(\prod_{ar} \theta(U_a^\mu) \cdot \theta(V_r^\mu) \right) \left(\prod_{ar} \theta(E^a) \cdot \theta(F^r) \right) \cdot e^{\alpha J^{a}_{\cdot} (\sum_r W^r)} \right] \right\rangle$$

⊙ Representing the step functions in integral form we get,

1). $$\theta(E^a) = \theta\left(N - \sum_i (J_i^a)^2\right) = \int_0^\infty dE_a \int \frac{d\hat{E}_a}{2\pi i} e^{-\hat{E}_a \left(E_a - 1 + \sum_i \frac{(J_i^a)^2}{N}\right)}$$

$$= \int_0^\infty \int \frac{dE_a \, d\hat{E}_a}{2\pi i} e^{-\hat{E}_a E_a + \hat{E}_a - \hat{E}_a \frac{\sum_i (J_i^a)^2}{N}}$$

2). $$\theta(F^r) = \theta\left(N - \sum_i (W_i^r)^2\right) = \int_0^\infty \int \frac{dF_r \, d\hat{F}_r}{2\pi i} e^{-\hat{F}_r F_r + \hat{F}_r - \hat{F}_r \sum_i \frac{(W_i^r)^2}{N}}$$

⊙ The disorder average ς in real case can be written as,

$$\varsigma = \prod_{a<b} \prod_{\gamma<\eta} \prod_{a\gamma} \prod_{a} \prod_{\gamma} \iint \frac{d Q_{ab}\, d\hat{Q}_{ab}}{2\pi} \iint \frac{d R_{\gamma\eta}\, d\hat{R}_{\gamma\eta}}{2\pi} \iint \frac{d P_{a\gamma}\, d\hat{P}_{a\gamma}}{2\pi} \iint \frac{d E_a\, d\hat{E}_a}{2\pi i} \iint \frac{d F_\gamma\, d\hat{F}_\gamma}{2\pi i} \left(\exp\left[-i \left(\sum_{a<b} \hat{Q}_{ab} Q_{ab} + \sum_{\gamma<\eta} R_{\gamma\eta} \hat{R}_{\gamma\eta} + \sum_{a\gamma} P_{a\gamma} \hat{P}_{a\gamma} \right) + \sum_a \left(-E_a \hat{E}_a + \hat{E}_a \right) + \sum_\gamma \left(-F_\gamma \hat{F}_\gamma + \hat{F}_\gamma \right) \right] \right)$$

$$\times \left[\left(\prod_a \int dJ^a \right) \left(\prod_\gamma \int dw^\gamma \right) \exp\left[\frac{i}{N} \left(\sum_{a<b} \hat{Q}_{ab} \sum_i J_i^a J_i^b + \sum_{\gamma<\eta} \hat{R}_{\gamma\eta} \sum_i w_i^\gamma w_i^\eta + \sum_{a\gamma} \hat{P}_{a\gamma} \sum_i J_i^a J_i^\gamma \right) + \left(\left(-\sum_a \hat{E}_a \right) \left(\sum_i (J_i^a)^2 \right) + \left(-\sum_\gamma \hat{F}_\gamma \sum_i (w^\gamma)^2 \right) \right) \right] \right]$$

$$\times \left\langle \prod_\mu \left[\prod_{a\gamma} \theta(U_a^\mu) \cdot \theta(V_\gamma^\mu) \right] \right\rangle_\xi e^{\eta \sum_{i\gamma} J_i^1 w_i^\gamma}$$

We bifurcate S into three terms.

$$S_1 : \exp\left[-\left(N\sum_{a<b} Q_{ab}\hat{Q}_{ab} + N\sum_{r<\eta} R_{r\eta}\hat{R}_{r\eta} + N\sum_{ar} P_{ar}\hat{P}_{ar} + \sum_a N\hat{E}_a(-E_a) + \sum_r N\hat{F}_r(-f_r)\right)\right]$$

$$S_2 : \exp\left[\sum_{a<b}\hat{Q}_{ab}\sum_i J_i^a J_i^b + \sum_{r<\eta}\hat{R}_{r\eta}\sum_i w_i^r w^\eta + \sum_{ar}\hat{P}_{ar}\sum_i J_i^a w_i^r - \sum_a \hat{E}_a\left(\sum_i (J_i^a)^2\right) - \sum_r \hat{F}_r\left(\sum_i (w_i^r)^2\right)\right]$$

$$S_3 : \left\langle \prod_\mu \left[\int \prod_{ar} \Theta(U_a^\mu)\,\Theta(V_r^\mu)\right]\right\rangle_{\xi} \cdot e^{n\sum_{ir} J_i^1 w_i^r}$$

$\hookrightarrow$ we've used: $\frac{i\hat{Q}_{ab}}{N} = \hat{Q}_{ab}$ / $\frac{i\hat{R}_{r\eta}}{N} = \hat{R}_{r\eta}$ / $\frac{i\hat{P}_{ar}}{N} = \hat{P}_{ar}$ / $\frac{i\hat{E}_a}{N} = E_a$ / $\frac{i\hat{F}_r}{N} = \hat{F}_r$

taking $\sum_i$ out, we get, for S_2;

$$S_2 : \exp\left[\sum_i\left[\sum_{a<b}\hat{Q}_{ab} J_i^a J_i^b + \sum_{r<\eta}\hat{R}_{r\eta} w_i^r w_i^\eta + \sum_{ar}\hat{P}_{ar} J_i^a w_i^r - \sum_a \hat{E}_a (J_i^a)^2 - \sum_r \hat{F}_r (w_i^r)^2\right]\right]$$

⊙ S_3 in this case would be same as Huang_etal

⊙ S_1 is relatively simple

⊙ S_2 is where we see non-trivial divergence in real & discrete space.

We'll now focus on S_2

$$S_2 : \exp\left(\sum_i \left[\hat{q}\sum_{a<b} J^a J^b + \hat{r}\sum_{\gamma<\eta} w^\gamma w^\eta + \frac{\hat{p}'}{2}\left(\left(\sum_a J^a + \sum_\gamma w^\gamma\right)^2 - \left(\sum J^a\right)^2 - \left(\sum w^\gamma\right)^2\right) + (\hat{p}-\hat{p}')\sum_\gamma J^\gamma w^\gamma - \hat{e}\sum_a (J^a)^2 - \hat{f}\sum_\gamma (w^\gamma)^2\right]\right)$$

$$\Rightarrow \left[\exp\left[\underbrace{\left(\sum J^a\right)^2\left(\frac{\hat{a}}{2}-\hat{p}'\right)}_{t_1} + \underbrace{\left(\sum w^\gamma\right)^2\left(\frac{\hat{r}}{2}-\hat{p}'\right)}_{t_2} + \underbrace{\left(\sum_a J^a + \sum_\gamma w^\gamma\right)^2\frac{\hat{p}'}{2}}_{t_3} - \sum_a (J^a)^2\left(\hat{e}+\hat{a}/2\right) - \sum_\gamma (w^\gamma)^2\left(\hat{f}+\hat{r}/2\right) + (\hat{p}-\hat{p}')\sum_\gamma J^\gamma w^\gamma\right]\right]^N$$

Using Hubbard-Stratonovich / Gaussian integral trick we get

$$\left\{ e^{-\frac{a}{2}\left(\sum x_i\right)^2} = \int \frac{dt}{\sqrt{2\pi}}\, e^{-t^2/2 + it\sqrt{a}\sum x_i} \right\}$$

$$\int dJ^a \int dw^r \exp\left\{ \left(\frac{\hat{a}}{2}-\hat{p}'\right)\left(\sum_a J^a\right)^2 - \left(\hat{e}+\frac{\hat{a}}{2}\right)\sum_a (J^a)^2 + (\hat{p}-\hat{p}')J'\sum_r w^r + \frac{\hat{p}'}{2}\left(\sum J^a + \sum_r w^r\right)^2 + \left(\frac{\hat{\gamma}}{2}-\hat{p}'\right)\left(\sum_r w^r\right)^2 - \left(\hat{j}+\frac{\hat{\gamma}}{2}\right)\sum_r (w^r)^2 \right\}$$

$$= \int Dz_1 \int Dz_3 \int dJ^a \int dw^r \exp\left\{ z_1\sqrt{\hat{a}-\hat{p}'}\sum_a J^a - \left(\frac{\hat{a}}{2}+\hat{e}\right)\sum_a (J^a)^2 + (\hat{p}-\hat{p}')J'\sum_r w^r + \left(\frac{\hat{\gamma}}{2}-\hat{p}'\right)\left(\sum_r w^r\right)^2 - \left(\hat{j}+\frac{\hat{\gamma}}{2}\right)\sum_r (w^r)^2 + z_3\sqrt{\hat{p}'}\left(\sum_a J^a + \sum_r w^r\right) \right\}$$

$$= \int Dz_1 \int Dz_3 \int dw^r \int dJ' \exp\left\{ \left(z_1\sqrt{\hat{a}-\hat{p}'}+z_3\sqrt{\hat{p}'}\right)J' - \left(\frac{\hat{a}}{2}+\hat{e}\right)(J')^2 + \left((\hat{p}-\hat{p}')J' + z_3\sqrt{\hat{p}'}\right)\sum_r w^r + \left(\frac{\hat{\gamma}}{2}-\hat{p}'\right)\left(\sum_r w^r\right)^2 - \left(\hat{j}+\frac{\hat{\gamma}}{2}\right)\sum_r (w^r)^2 \right\}$$

$$\times \int dJ^{a>1} \exp\left\{ \left(z_1\sqrt{\hat{a}-\hat{p}'}+z_3\sqrt{\hat{p}'}\right)\sum_{a>1} J^a - \left(\frac{\hat{a}}{2}+\hat{e}\right)\sum_{a>1}(J^a)^2 \right\}$$

$$\ldots \left[\hat{a} = z_1\sqrt{\hat{a}-\hat{p}'} + z_3\sqrt{\hat{p}'} \right]$$

$$= \int Dz_1 \int Dz_3 \int dw^r \int dJ^1 \exp\left\{ \hat{a}J^1 - \left(\hat{e} + \frac{\hat{a}}{2}\right)(J^1)^2 + \left[\left(\hat{p} - \hat{p}'\right) J^1 + z_3 \sqrt{\hat{p}'}\right] \sum_r w^r + \left(\frac{\hat{r} - \hat{p}'}{2}\right) \cdot \left(\sum_r w^r\right)^2 - \left(\frac{\hat{r}}{2} + \hat{f}\right) \sum_r (w^r)^2 \right\} \times$$

$$\int dJ^{a>1} \exp\left\{ \hat{a} \sum_{a>1} J^a - \left(\hat{e} + \frac{\hat{a}}{2}\right)\left(\sum_{a>1} (J^a)^2\right) \right\}$$

$$= \int Dz_1 \int Dz_3 \int dw^r \exp\left\{ z_3 \sqrt{\hat{p}'} \sum_r w^r + \left(\frac{\hat{r} - \hat{p}'}{2}\right)\left(\sum_r w^r\right)^2 - \left(\frac{\hat{r}}{2} + \hat{f}\right) \cdot \sum_r (w^r)^2 \right\} .$$

$$\times \int dJ^1 \exp\left\{ - \left(\hat{e} + \frac{\hat{a}}{2}\right)(J^1)^2 + \left(\hat{a} + \left(\hat{p} - \hat{p}'\right) \sum_r w^r\right) J^1 \right\} .$$

$$\times \left(\sqrt{\frac{2\pi}{2\hat{e} + \hat{a}}} \cdot \exp\left(\frac{\hat{a}^2}{2(2\hat{e} + \hat{a})}\right)\right)^{n-1} .$$

$$= \int Dz_1 \int Dz_3 \int dw^r \exp\left\{ z_3 \sqrt{\hat{p}'} \sum_r w^r + \left(\frac{\hat{r} - \hat{p}'}{2}\right)\left(\sum_r w^r\right)^2 - \left(\frac{\hat{r}}{2} + \hat{f}\right) \sum_r (w^r)^2 + \frac{\left(\hat{a} + (\hat{p} - \hat{p}') \sum_r w^r\right)^2}{2(2\hat{e} + \hat{a})} \right\} \times \left(\sqrt{\frac{2\pi}{2\hat{e} + \hat{a}}}\right)^{n} \left(\exp\left\{\frac{\hat{a}^2}{2(2\hat{e} + \hat{a})}\right\}\right)^{n-1}$$

$$= \int Dz_1 \int Dz_3 \int dw^r \exp\left\{ z_3 \sqrt{\hat{p}'} \sum_r w^r + \left(\frac{\hat{r} - \hat{p}'}{2}\right) \cdot \left(\sum_r w^r\right)^2 - \left(\frac{\hat{r}}{2} + \hat{f}\right) \sum_r (w^r)^2 + \frac{\hat{a}(\hat{p} - \hat{p}')}{(2\hat{e} + \hat{a})} \sum_r w^r + \frac{(\hat{p} - \hat{p}')^2 \left(\sum_r w^r\right)^2}{2(2\hat{e} + \hat{a})} \right\} \times \left(\sqrt{\frac{2\pi}{2\hat{e} + \hat{a}}} \cdot \exp\left(\frac{\hat{a}^2}{2(2\hat{e} + \hat{a})}\right)\right)^{n}.$$

$$= \int Dz_1 \int Dz_3 \int dw^r \exp\left\{ \left[z_3 \sqrt{\hat{p}'} + \frac{\hat{a}(\hat{p} - \hat{p}')}{2\hat{e} + \hat{a}}\right] \sum_r w^r + \frac{1}{2}\left[\hat{r} - \hat{p}' + \frac{(\hat{p} - \hat{p}')^2}{2\hat{e} + \hat{a}}\right] \left(\sum_r w^r\right)^2 - \left(\frac{\hat{r}}{2} + \hat{f}\right) \sum_r (w^r)^2 \right\}.$$

$$\times \left(\sqrt{\frac{2\pi}{2\hat{e} + \hat{a}}} \exp\left(\frac{\hat{a}^2}{2(2\hat{e} + \hat{a})}\right)\right)^{n}.$$

$$\int Dz_1 \int Dz_2 \int Dz_3 \cdot \int dw^r \cdot \exp\left\{ \left[z_3\sqrt{\hat{p}'} + z_2 \sqrt{\hat{r} - \hat{p} + \frac{(\hat{p}-\hat{p}')^2}{(2\hat{e}+\hat{a})}} + \frac{\hat{a}\,(\hat{p}-\hat{p}')}{(2\hat{e}+\hat{a})} \right] \sum_r w^r - \left(\frac{\hat{r}}{2}+\hat{j}\right) \sum_r (w^r)^2 \right\}$$

$$\times \left(\sqrt{\frac{2\pi}{2\hat{e}+\hat{a}}} \exp\left(\frac{\hat{a}^2}{2(2\hat{e}+\hat{a})}\right)\right)^n .$$

$$\ldots \hat{a}' = z_3 \sqrt{\hat{p}'} + z_2 \sqrt{\hat{r} - \hat{p} + \frac{(\hat{p}-\hat{p}')^2}{2\hat{e}+\hat{a}}}$$

$$\int Dz \left[\int dw \cdot \exp\left(\left[\hat{a}' + \frac{\hat{a}\,(\hat{p}-\hat{p}')}{2\hat{e}+\hat{a}} \right] w - \left(\frac{\hat{r}+2\hat{j}}{2}\right) \cdot w^2 \right) \right]^n \times \left(\sqrt{\frac{2\pi}{2\hat{e}+\hat{a}}} \cdot \exp\left(\frac{\hat{a}^2}{2(2\hat{e}+\hat{a})}\right)\right)^n$$

$$\Rightarrow \left[\sqrt{\frac{2\pi}{2\hat{j}+\hat{r}}} \cdot \exp\left(\frac{\left[\hat{a}' + \frac{\hat{a}\,(\hat{p}-\hat{p}')}{2\hat{e}+\hat{a}} \right]^2}{2\,(2\hat{j}+\hat{r})} \right) \right]^n \cdot \left(\sqrt{\frac{2\pi}{2\hat{e}+\hat{a}}} \cdot \exp\left(\frac{\hat{a}^2}{2(2\hat{e}+\hat{a})}\right)\right)^n$$

$$\Rightarrow \left(\sqrt{\frac{2\pi}{2\hat{f}+\hat{r}}}\right)^{m} \left(\sqrt{\frac{\pi}{2\hat{e}+\hat{q}}}\right)^{n} \cdot \int Dz_1 \int Dz_2 \int Dz_3 \cdot \exp\left(\frac{n \cdot a^2}{2(2\hat{e}+\hat{q})}\right) \cdot \exp\left(m \frac{\left(\hat{a}' + \hat{a}\frac{(\hat{p}-\hat{p}')}{2\hat{e}+\hat{q}}\right)^2}{2(2\hat{f}+\hat{r})}\right) \quad \text{(A)}$$

$$\ldots \quad \hat{a} = z_1\sqrt{\hat{q}-\hat{p}'} + z_3\sqrt{\hat{p}'}$$

$$\ldots \quad \hat{a}' = z_3\sqrt{\hat{p}'} + z_2\sqrt{\hat{r}-\hat{p}' + \frac{(\hat{p}-\hat{p}')^2}{2\hat{e}+\hat{q}}}$$

$$\Rightarrow \frac{\partial \log\left[\int Dz \; A\right]}{\partial m}\Bigg|_{m=n=0} = \frac{\int Dz \; \partial A/\partial m \,\big|_{m=n=0}}{\int Dz \; A \,\big|_{m=n=0}} = 1.$$

$$\int Dz \frac{\partial A}{\partial m}\Bigg|_{m=n=0} = \int Dz \left(\hat{a}' + \hat{a}\frac{(\hat{p}-\hat{p}')}{2\hat{e}+\hat{a}}\right)^2 \times \frac{1}{2(2\hat{f}+\hat{r})}.$$

$$= \left(\cancel{\hat{p}'} + \hat{r} - \cancel{\hat{p}'} + \frac{(\hat{p}-\hat{p}')^2}{2\hat{e}+\hat{q}} + \left(\frac{\hat{p}-\hat{p}'}{2\hat{e}+\hat{q}}\right)^2\left[\hat{q} - \hat{p}' + \hat{p}'\right] + 2\cdot\hat{p}'\cdot\left(\frac{\hat{p}-\hat{p}'}{2\hat{e}+\hat{q}}\right)\right) \times \frac{1}{2(2\hat{f}+\hat{r})}$$

$$= \hat{\gamma} + \frac{(\hat{p} - \hat{p}')^2}{2\hat{e} + \hat{q}} + \hat{q}\left(\frac{\hat{p} - \hat{p}'}{2\hat{e} + \hat{q}}\right)^2 + 2 \cdot \hat{p}' \cdot \left(\frac{\hat{p} - \hat{p}'}{2\hat{e} + \hat{q}}\right)$$

$$= \hat{\gamma} + \frac{\hat{p}^2 - \hat{p}'^2}{2\hat{e} + \hat{q}} + \frac{\hat{q}\,(\hat{p} - \hat{p}')^2}{(2\hat{e} + \hat{q})^2}.$$

$$= \hat{\gamma} + \frac{(\cancel{\hat{q}\,\hat{p}^2} - \cancel{\hat{q}\,\hat{p}'^2} + 2\hat{e} \cdot \hat{p}^2 - 2\hat{e} \cdot \hat{p}'^2 + \cancel{\hat{q}\,\hat{p}^2} + \cancel{\hat{q}\,\hat{p}'^2} - 2\hat{q} \cdot \hat{p} \cdot \hat{p}')}{(2\hat{e} + \hat{q})^2}$$

$$= \hat{\gamma} + \frac{(2\hat{q}\,\hat{p}^2 + 2\hat{e}\,\hat{p}^2 - 2\hat{e}\,\hat{p}'^2 - 2\hat{q}\,\hat{p}\,\hat{p}')}{(2\hat{e} + \hat{q})^2}$$

$$= \hat{\gamma} + \frac{(\hat{p} - \hat{p}')(2\hat{q}\,\hat{p} + 2\hat{e}\,(\hat{p} + \hat{p}'))}{(2\hat{e} + \hat{q})^2}.$$

thus, $$\left.\frac{\partial S_2}{\partial m}\right|_{m=n=0} \Rightarrow \frac{\partial\left(-\frac{m}{2}\log\left(\frac{2\hat{f}+\hat{r}}{2\pi}\right)-\frac{n}{2}\log\left(\frac{2\hat{e}+\hat{v}}{2\pi}\right)\right)}{\partial m = n = 0} + \frac{1}{2(2\hat{f}+\hat{r})} \times \left[\hat{r}+\frac{(\hat{p}-\hat{p}')(2\hat{v}\hat{f}+2\hat{e}(\hat{p}+\hat{p}'))}{(2\hat{e}+\hat{v})^2}\right]$$

$$\left.\frac{\partial S_2}{\partial m}\right|_{m=n=0} \Rightarrow -\frac{1}{2}\log\left(2\hat{f}+\hat{r}\right) + \frac{1}{2(2\hat{f}+\hat{r})} \times \left[\hat{r}+\frac{(\hat{p}-\hat{p}')\left(2\hat{v}\hat{f}+2\hat{e}(\hat{p}+\hat{p}')\right)}{(2\hat{e}+\hat{v})^2}\right] + \log(2\pi)$$

$$\lim_{N\to\infty} \frac{\partial S_1}{\partial m}\bigg|_{m=n=0} = \frac{\partial}{\partial m}\left(\frac{n(n-1)}{2} q\hat{q} + \frac{m(m-1)}{2} r\hat{r} + m p\hat{p} + (n-1) m p'\hat{p}' + n\hat{e}(1-e) + m(1-f)\cdot\hat{f}\right)\frac{x - N}{N}$$

$$= -1\left[-\frac{r\hat{r}}{2} + p\hat{p} - p'\hat{p}' + (1-f)\hat{f} - qp\right]$$

$$= \frac{r\hat{r}}{2} + p'\hat{p}' - p\hat{p} + qp - \hat{f}(1-f).$$

$$\lim_{N\to\infty} \frac{\partial S_3}{\partial m}\bigg|_{m=n=0} = \alpha\int Dt\, H^{-1}(\tilde{t})\int Dw \int_{\tilde{t}}^{\infty} Dy \log\left(H(h(w,t,y))\right)$$

$$\therefore\ f(x) = \frac{r\hat{r}}{2} + p'\hat{p}' - p\hat{p} + xp - \hat{f}(1-f) - \frac{1}{2}\log\left(2\hat{f}+\hat{r}\right)$$

$$+ \frac{1}{2}\cdot\frac{1}{(2\hat{f}+\hat{r})} * \left[\hat{r} + \frac{(\hat{p}-\hat{p}')\left(2\hat{q}\hat{p} + 2\hat{e}(\hat{p}+\hat{p}')\right)}{(2\hat{e}+\hat{q})^2}\right].$$

$$+ \alpha \int Dt\, H^{-1}(\tilde{t}) \int Dw \int_{\tilde{t}}^{\infty} Dy\ \log\left(H(h(w,t,y))\right)$$

- $H(x) = \int_x^{\infty} Dz$
- $V_w = r - \frac{p^2}{q} - \frac{(p-p')^2}{1-q}$
- $\tilde{t} = -\sqrt{\frac{q}{1-q}}\, t$
- $Dz = G(z)\, dz$
- $G(z) = \exp(-z^2/2)/\sqrt{2\pi}$
- $h(w,t,y) = \frac{-1}{\sqrt{1-r}}\left[\frac{(p-p')y}{\sqrt{1-q}} + \sqrt{V_w}\cdot w + \frac{p't}{\sqrt{q}}\right]$

We know that $\frac{\partial f}{\partial q}, \frac{\partial f}{\partial \hat{q}} = 0$ cannot depend on new order parameters

- 1). $\hat{q} = \frac{q}{(1-q)^2}$
- 2). $\frac{q^2}{1-q} = \alpha \int Dt\, R^2\left(\frac{\sqrt{q}}{\sqrt{1-q}}\, t\right)$.

- the following cases are similar to Huong et al.

- 3). $\hat{p} = m + \frac{\alpha}{\sqrt{(1-q)(1-r)}} \int Dw \int Dt\, R(\tilde{t})\, R\left(h(w,t,y=\tilde{t})\right)$
- 4). $\hat{p}' = \frac{\alpha}{\sqrt{(1-q)(1-r)}} \int Dw \int Dt\, H^{-1}(\tilde{t})\, R(\tilde{t}) \int_{\tilde{t}}^{\infty} Dy\, R(h)$.
- 5). $\hat{r} = \frac{1}{1-r} \int Dt\, H^{-1}(\tilde{t}) \int Dw \int_{\tilde{t}}^{\infty} Dy\, R^2(h)$.

but we also want $\partial \hat{r}, \partial \hat{b}, \partial \hat{b}', \partial \vec{f}, \partial f, \partial \vec{e}, \partial e$.

6). $\dfrac{\partial f(n)}{\partial \hat{r}} = 0 \Rightarrow \dfrac{r}{2} - \dfrac{1}{2(2\vec{f}+\hat{r})} + \dfrac{1}{2(2\vec{f}+\hat{r})} \cdot 1 + -\dfrac{1}{2(2\vec{f}+\hat{r})^2}\left\{\hat{r} + (\hat{b}-\hat{b}') \times \dfrac{(2\hat{a}\hat{b}+2\vec{e}(\hat{b}+\hat{b}')}{(2\vec{e}+\hat{a})^2}\right.$

$$\Rightarrow r = \frac{1}{(2\vec{f}+\hat{r})^2}\left[\hat{r} + (\hat{b}-\hat{b}') \cdot \frac{(2\hat{a}\hat{b}+2\vec{e}(\hat{b}+\hat{b}'))}{(2\vec{e}+\hat{a})^2}\right].$$

7). $\dfrac{\partial f(n)}{\partial f} \Rightarrow -\vec{f} = 0 \Rightarrow \vec{f} = 0.$

8). $\dfrac{\partial f(n)}{\partial \vec{e}} = 0 \Rightarrow \vec{e} = \dfrac{\hat{a}(\hat{b}' - 3\hat{b})}{2(\hat{b}+\hat{b}')}$

9). $\dfrac{\partial f(n)}{\partial \vec{f}} = 0 \Rightarrow f = 1 + \dfrac{1}{(2\vec{f}+\hat{r})} + \dfrac{1}{(2\vec{f}+\hat{r})^2}\left[\hat{r} + (\hat{b}-\hat{b}')\dfrac{(2\hat{a}\hat{b}+2\vec{e}(\hat{b}+\hat{b}'))}{(2\vec{e}+\hat{a})^2}\right].$

10). $\dfrac{d\, f(x)}{d\, \hat{p}} = 0 \Rightarrow p = \dfrac{2\hat{f}\,(\hat{a}+\hat{e}) - \hat{a}\,\hat{p}'}{(2\hat{f}+\hat{r})\,(2\hat{e}+\hat{a})^{2}}$.

11). $\dfrac{d\, f(x)}{d\, \hat{p}'} = 0 \Rightarrow p' = \dfrac{\hat{a}\,\hat{p} + 2\hat{e}\,\hat{p}'}{(2\hat{f}+\hat{r})\,(2\hat{e}+\hat{a})^{2}}$