Notes for sparse perceptron

1 Observation

Consider a "regular" perceptron problem:

$$\hat{y} = \text{sign}\left(\sum_i J_i x_i^\mu\right) \qquad , \qquad \{y^\mu \longleftrightarrow x^\mu\}. \tag{1}$$

The capacity α_c was computed by Gardner (1988). If a random subset of ϕN synapses are set to 0, the capacity drops to $\phi\alpha_c$. However, if the weights J_i are learned without the sparsity constraint, and then the smallest weights (in absolute value) are set to 0, the capacity is close to α_c. We develop a theory for this sparse perceptron.

2 Idea

Suppose J is a dense solution to a problem, and suppose for simplicity that the y^μ's are equally likely to be 0 or 1. We know the distribution of the entries J_i,

$$P(J) = \frac{1}{\sqrt{2\pi}} e^{-\frac{J^2}{2}} \equiv G(J). \tag{2}$$

The mean is 0 because of the symmetry $P(y^\mu = +1) = P(y^\mu = -1)$, and the variance is 1 becasue of the normalization condition $\sum_i J_i^2 = N$. Denote by w the weight vector equal to J for entries larger (in absolute value) than the ϕN largest entry, and 0 otherwise. We can compute the overlap of J and w:

$$\begin{aligned}
\frac{1}{N}\sum_i J_i w_i &= \int dJ J^2 G(J)\, \Theta\left(|J| > \Phi^{-1}(\phi)\right) \\
&= 2\int_{\Phi^{-1}(\phi)}^{\infty} dJ J^2 G(J) \\
&= 2\left[\Phi^{-1}(\phi)\, G\left(\Phi^{-1}(\phi)\right) + H\left(\Phi^{-1}(\phi)\right)\right],
\end{aligned} \tag{3}$$

where $H(x) = \int_x^\infty G(y)\, dy$, and $\Phi^{-1}(x)$ is the inverse cumulative distribution function of a Gaussian.
We are interested in understanding whether w is also a solution to the problem, i.e.,

$$y^\mu = \text{sign}\left(\sum_i w_i x_i^\mu\right). \tag{4}$$

A similar problem was analyzed by Huang and Kabashima (2014). They considered a perceptron where the weights are binary $J_i \in \{+1, -1\}$. In their case, w was not sparser relative to J, but it had a specific overlap, in order to study whether solutions to a perceptron are isolated or belong to a dense region. We will assume that the classification capabilities of w depend only on its overlap with J, and not on the direction in weight space from J to w. We will also assume that the variance of w_i is 1 (i.e., that decrease in variance due to setting the smallest weights to 0 is negligible). Hence the distribution of w is also $G(w)$ (but J and w are highly correlated such that their overlap is given by the formula above).

3 Franz-Parisi Potential

We follow [HK14] in their computation of the Franz-Parisi potential, which enumerates pairs of vectors J, w which both solve the problem. We make the necessary modifications, allowing J, w to be continuous instead of binary. The free energy of configuration w relative to the reference configuration J:

$$F(x) = \lim_{\substack{n\to 0\\ m\to 0}} \frac{\partial}{\partial m}\left\langle \sum_{\{\mathbf{J}^a,\mathbf{w}^\gamma\}} \prod_\mu \left[\prod_{a,\gamma}\Theta\left(u_a^\mu\right)\Theta\left(v_\gamma^\mu\right)\right]\exp\left(x\sum_{\gamma,i}J_i^1 w_i^\gamma\right)\right\rangle. \tag{5}$$

Here x is the interaction strength between J and w. Note that J and w have different sets of replicas, indexed by $a, b = 1, \ldots, n$ and $\gamma, \eta = 1, \ldots, m$ respectively.
We define auxilary variables and order parameters

$$\begin{aligned}
u_a^\mu &= \frac{1}{\sqrt{N}}\sum_i J_i^a \xi_i^\mu \\
v_\gamma^\mu &= \frac{1}{\sqrt{N}}\sum_i w_i^\gamma \xi_i^\mu \\
Q_{ab} &= \frac{1}{N}\sum_i J_i^a J_i^b = q\left(1-\delta_{ab}\right)+\delta_{ab} \qquad \text{(correct also for continuous, normalized } J_i\text{'s)} \\
P_{a\gamma} &= \frac{1}{N}\sum_i J_i^a w_i^\gamma = p\delta_{a1}+p'\left(1-\delta_{a1}\right) \\
R_{\gamma\eta} &= \frac{1}{N}\sum_i w_i^\gamma w_i^\eta = r\left(1-\delta_{\gamma\eta}\right)+\delta_{\gamma\eta} \qquad \text{(correct also for continuous, normalized } w_i\text{'s)}
\end{aligned} \tag{6}$$

The physical interpretation of the order parameters is the overlap of different sets of weights:

$$\begin{aligned}
\left\langle u_a^\mu u_b^\mu\right\rangle &= \left\langle \frac{1}{\sqrt{N}}\sum_i J_i^a \xi_i^\mu \frac{1}{\sqrt{N}}\sum_j J_j^b \xi_j^\mu\right\rangle \\
&= \frac{1}{N}\sum_{i,j} J_i^a J_j^b \left\langle \xi_i^\mu \xi_j^\mu\right\rangle \\
&= \frac{1}{N}\sum_{i,j} J_i^a J_j^b \delta_{ij} \\
&= Q_{ab} \\
\left\langle u_a^\mu v_\gamma^\mu\right\rangle &= P_{a\gamma} \\
\left\langle v_\gamma^\mu v_\eta^\mu\right\rangle &= R_{\gamma\eta}
\end{aligned} \tag{7}$$

Computation of the free energy by introducing order parameters and their conjugate variables,

$$\begin{aligned}
\mathcal{S} &= \prod_{a<b}\prod_{\gamma<\eta}\prod_{a,\gamma}\int \mathrm{d}Q_{ab}\int \mathrm{d}P_{a\gamma}\int \mathrm{d}R_{\gamma\eta}\sum_{\{\mathbf{J}^a,\mathbf{w}^\gamma\}}\left\langle \prod_\mu \left[\prod_{a,\gamma}\Theta\left(u_a^\mu\right)\Theta\left(v_\gamma^\mu\right)\right]\exp\left(x\sum_{\gamma,i}J_i^1 w_i^\gamma\right)\right\rangle \\
&\quad\times\delta\left(Q_{ab}-\frac{1}{N}\sum_i J_i^a J_i^b\right)\delta\left(P_{a\gamma}-\frac{1}{N}\sum_i J_i^a w_i^\gamma\right)\delta\left(R_{\gamma\eta}-\frac{1}{N}\sum_i w_i^\gamma w_i^\eta\right) \\
&= \prod_{a<b}\prod_{\gamma<\eta}\prod_{a,\gamma}\int \mathrm{d}Q_{ab}\int \mathrm{d}P_{a\gamma}\int \mathrm{d}R_{\gamma\eta}\sum_{\{\mathbf{J}^a,\mathbf{w}^\gamma\}}\left\langle \prod_\mu \left[\prod_{a,\gamma}\Theta\left(u_a^\mu\right)\Theta\left(v_\gamma^\mu\right)\right]\exp\left(x\sum_{\gamma,i}J_i^1 w_i^\gamma\right)\right\rangle
\end{aligned}$$

$$
\begin{aligned}
&\times \exp\left\{-i\left[\sum_{a<b}\hat{Q}_{ab}\left(Q_{ab}-\frac{1}{N}\sum_i J_i^a J_i^b\right)+\sum_{a,\gamma}\hat{P}_{a\gamma}\left(P_{a\gamma}-\frac{1}{N}\sum_i J_i^a w_i^\gamma\right)+\sum_{\gamma<\eta}\hat{R}_{\gamma\eta}\left(R_{\gamma\eta}-\frac{1}{N}\sum_i w_i^\gamma w_i^\eta\right)\right]\right\}\\
=\ &\prod_{a<b}\prod_{\gamma<\eta}\prod_{a,\gamma}\int \mathrm{d}Q_{ab}\int \mathrm{d}P_{a\gamma}\int \mathrm{d}R_{\gamma\eta}\sum_{\{\mathbf{J}^a,\mathbf{w}^\gamma\}}\left\langle\prod_\mu\left[\prod_{a,\gamma}\Theta\left(u_a^\mu\right)\Theta\left(v_\gamma^\mu\right)\right]\exp\left(x\sum_{\gamma,i}J_i^1 w_i^\gamma\right)\right\rangle\\
&\times \exp\left\{-i\left(\sum_{a<b}\hat{Q}_{ab}Q_{ab}+\sum_{a,\gamma}\hat{P}_{a\gamma}P_{a\gamma}+\sum_{\gamma<\eta}\hat{R}_{\gamma\eta}R_{\gamma\eta}\right)\right\}\\
&\times \exp\left\{\frac{i}{N}\left(\sum_{a<b}\hat{Q}_{ab}\sum_i J_i^a J_i^b+\sum_{a,\gamma}\hat{P}_{a\gamma}\sum_i J_i^a w_i^\gamma+\sum_{\gamma<\eta}\hat{R}_{\gamma\eta}\sum_i w_i^\gamma w_i^\eta\right)\right\}\\
=\ &\prod_{a<b}\prod_{\gamma<\eta}\prod_{a,\gamma}\int \mathrm{d}Q_{ab}\int \mathrm{d}P_{a\gamma}\int \mathrm{d}R_{\gamma\eta}\sum_{\{\mathbf{J}^a,\mathbf{w}^\gamma\}}\left\langle\prod_\mu\left[\prod_{a,\gamma}\Theta\left(u_a^\mu\right)\Theta\left(v_\gamma^\mu\right)\right]\right\rangle\exp\left(x\sum_{\gamma,i}J_i^1 w_i^\gamma\right)\\
&\times \exp\left\{-N\left(\sum_{a<b}\hat{Q}_{ab}Q_{ab}+\sum_{a,\gamma}\hat{P}_{a\gamma}P_{a\gamma}+\sum_{\gamma<\eta}\hat{R}_{\gamma\eta}R_{\gamma\eta}\right)\right\}\\
&\times \exp\left(\sum_{a<b}\hat{Q}_{ab}\sum_i J_i^a J_i^b+\sum_{a,\gamma}\hat{P}_{a\gamma}\sum_i J_i^a w_i^\gamma+\sum_{\gamma<\eta}\hat{R}_{\gamma\eta}\sum_i w_i^\gamma w_i^\eta\right). \qquad (8)
\end{aligned}
$$

The following sums over order parameters (under the replica symmetry ansatz) and weights are useful,

$$
\begin{aligned}
\sum_{a<b}\hat{Q}_{ab}Q_{ab} &= \frac{1}{2}n\left(n-1\right)q\hat{q}\\
\sum_{a,\gamma}\hat{P}_{a\gamma}P_{a\gamma} &= \sum_\gamma \hat{P}_{1\gamma}P_{1\gamma}+\sum_{a>1,\gamma}\hat{P}_{a\gamma}P_{a\gamma}\\
&= mp\hat{p}+\left(n-1\right)mp'\hat{p}'\\
\sum_{\gamma<\eta}\hat{R}_{\gamma\eta}R_{\gamma\eta} &= \frac{1}{2}m\left(m-1\right)r\hat{r}\\
\sum_i\sum_{a<b}\hat{Q}_{ab}J_i^a J_i^b &= \sum_i\left(\hat{q}\sum_{a<b}J_i^a J_i^b\right)\\
&= \sum_i\frac{1}{2}\hat{q}\left[\left(\sum_a J_i^a\right)^2-\sum_a\left(J_i^a\right)^2\right]\\
&= \frac{N}{2}\hat{q}\left[\left(\sum_a J^a\right)^2-\sum_a\left(J^a\right)^2\right]\\
\sum_i\sum_{a,\gamma}\hat{P}_{a\gamma}J_i^a w_i^\gamma &= \sum_i\left(\left(\hat{p}-\hat{p}'\right)J_i^1\sum_\gamma w_i^\gamma+\hat{p}'\sum_{a,\gamma}J_i^a w_i^\gamma\right)\\
&= \sum_i\left\{\left(\hat{p}-\hat{p}'\right)J_i^1\sum_\gamma w_i^\gamma+\frac{1}{2}\hat{p}'\left[\left(\sum_a J_i^a+\sum_\gamma w_i^\gamma\right)^2-\left(\sum_a J_i^a\right)^2-\left(\sum_\gamma w_i^\gamma\right)^2\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= N\left\{\left(\hat{p}-\hat{p}'\right)J^1\sum_\gamma w^\gamma+\frac{1}{2}\hat{p}'\left[\left(\sum_a J^a+\sum_\gamma w^\gamma\right)^2-\left(\sum_a J^a\right)^2-\left(\sum_\gamma w^\gamma\right)^2\right]\right\} \\
\sum_i\sum_{\gamma<\eta}\hat{R}_{\gamma\eta}w_i^\gamma w_i^\eta &= \frac{N}{2}\hat{r}\left[\left(\sum_\gamma w^\gamma\right)^2-\sum_\gamma\left(w^\gamma\right)^2\right]
\end{aligned} \tag{9}
$$

To compute $\left\langle\prod_\mu\left[\prod_{a,\gamma}\Theta\left(u_a^\mu\right)\Theta\left(v_\gamma^\mu\right)\right]\right\rangle$ consider the substitutions

$$
\begin{aligned}
u_a &= \sqrt{1-q}y_a+\sqrt{q}t \\
v_\gamma &= \sqrt{1-r}y'_\gamma+\frac{p-p'}{\sqrt{1-q}}y_1+\sqrt{v_\omega}\omega+\frac{p'}{\sqrt{q}}t \\
v_\omega &= r-\frac{p'^2}{q}-\frac{\left(p-p'\right)^2}{1-q}
\end{aligned} \tag{10}
$$

with $t,\omega,\{y_a\}_{a=1,\ldots,n},\{y'_\gamma\}_{\gamma=1,\ldots,m}$ standard Gaussian variables. We check that the substitutions preserve the averages over products of u,v:

$$
\begin{aligned}
\langle u_a u_b\rangle &= \left\langle\left(\sqrt{1-q}y_a+\sqrt{q}t\right)\left(\sqrt{1-q}y_b+\sqrt{q}t\right)\right\rangle \\
&= (1-q)\,\delta_{ab}+q \\
&= Q_{ab} \\
\langle u_a v_\gamma\rangle &= \left\langle\left(\sqrt{1-q}y_a+\sqrt{q}t\right)\left(\sqrt{1-r}y'_\gamma+\frac{p-p'}{\sqrt{1-q}}y_1+\sqrt{v_\omega}\omega+\frac{p'}{\sqrt{q}}t\right)\right\rangle \\
&= \sqrt{1-q}\frac{p-p'}{\sqrt{1-q}}\delta_{a1}+\sqrt{q}\frac{p'}{\sqrt{q}} \\
&= \left(p-p'\right)\delta_{a1}+p' \\
&= P_{a\gamma} \\
\langle v_\gamma v_\eta\rangle &= \left\langle\left(\sqrt{1-r}y'_\gamma+\frac{p-p'}{\sqrt{1-q}}y_1+\sqrt{v_\omega}\omega+\frac{p'}{\sqrt{q}}t\right)\left(\sqrt{1-r}y'_\eta+\frac{p-p'}{\sqrt{1-q}}y_1+\sqrt{v_\omega}\omega+\frac{p'}{\sqrt{q}}t\right)\right\rangle \\
&= (1-r)\,\delta_{\gamma\eta}+\frac{\left(p-p'\right)^2}{1-q}+v_\omega+\frac{p'^2}{q} \\
&= (1-r)\,\delta_{\gamma\eta}+\frac{\left(p-p'\right)^2}{1-q}+\frac{p'^2}{q}+r-\frac{p'^2}{q}-\frac{\left(p-p'\right)^2}{1-q} \\
&= (1-r)\,\delta_{\gamma\eta}+r \\
&= R_{\gamma\eta}
\end{aligned} \tag{11}
$$

The average over the input-output association statistics is

$$
\begin{aligned}
\left\langle\prod_\mu\left[\prod_{a,\gamma}\Theta\left(u_a^\mu\right)\Theta\left(v_\gamma^\mu\right)\right]\right\rangle &= \left\langle\prod_\mu\left[\prod_{a,\gamma}\int_0^\infty dw_a\int\frac{dx_a}{2\pi}e^{ix_a\left(w_a-u_a^\mu\right)}\int_0^\infty dw'_\gamma\int\frac{dx'_\gamma}{2\pi}e^{ix'_\gamma\left(w'_\gamma-v_\gamma^\mu\right)}\right]\right\rangle \\
&= \left[\prod_{a,\gamma}\int_0^\infty dw_a\int_0^\infty dw'_\gamma\int\frac{dx_a}{2\pi}\int\frac{dx'_\gamma}{2\pi}e^{i\left(x_aw_a+x'_\gamma w'_\gamma\right)}\left\langle\exp\left(-i\left(\sum_a x_au_a+\sum_\gamma x'_\gamma v_\gamma\right)\right)\right\rangle\right]^{\alpha N} \\
\left\langle\exp\left(-i\left(\sum_a x_au_a+\sum_\gamma x'_\gamma v_\gamma\right)\right)\right\rangle &= \prod_{a,\gamma}\int Dt\int D\omega\int Dy_a\int Dy'_\gamma\exp\left\{-i\left[\sum_a x_a\left(\sqrt{1-q}y_a+\sqrt{q}t\right)+\sum_\gamma x'_\gamma\left(\sqrt{1-r}y'_\gamma+\frac{p-p'}{\sqrt{1-q}}y_1+\sqrt{v_\omega}\omega+\frac{p'}{\sqrt{q}}t\right)\right]\right\}
\end{aligned} \tag{12}
$$

Some useful integrals and definitions to compute this average

$$
\begin{aligned}
\int \prod_{\gamma} Dy'_{\gamma} \exp\left(-i\sqrt{1-r}\sum_{\gamma} x'_{\gamma} y'_{\gamma}\right) &= \int \prod_{\gamma} \frac{dy'_{\gamma}}{\sqrt{2\pi}} \exp\left(\sum_{\gamma}\left(-\frac{1}{2}y'^2_{\gamma} - i\sqrt{1-r}x'_{\gamma}y'_{\gamma}\right)\right) \\
&= \prod_{\gamma} \exp\left(-\frac{1-r}{2}x'^2_{\gamma}\right) \\
\int \prod_{a>1} Dy_a \exp\left(-i\sqrt{1-q}\sum_a x_a y_a\right) &= \prod_{a>1} \exp\left(-\frac{1-q}{2}x_a^2\right) \\
\tilde{t} &= -\frac{\sqrt{q}t}{\sqrt{1-q}} \\
h(\omega,t,y) &= -\frac{1}{\sqrt{1-r}}\left[\frac{p-p'}{\sqrt{1-q}}y + \sqrt{v_\omega}\omega + \frac{p'}{\sqrt{q}}t\right] \qquad (13)
\end{aligned}
$$

We define the integral $\mathcal{I}_1$ and carry out the computation.

$$
\begin{aligned}
\mathcal{I}_1 &= \prod_{a,\gamma} \int_0^\infty dw_a \int_0^\infty dw'_\gamma \int \frac{dx_a}{2\pi} \int \frac{dx'_\gamma}{2\pi} \int Dt \int D\omega \int Dy_a \int Dy'_\gamma \exp\left\{-i\left[\sum_a x_a\left(\sqrt{1-q}y_a + \sqrt{q}t - w_a\right) + \sum_\gamma x'_\gamma\left(\sqrt{1-r}y'_\gamma + \frac{p-p'}{\sqrt{1-q}}y_1 + \sqrt{v_\omega}\omega + \frac{p'}{\sqrt{q}}t - w'_\gamma\right)\right]\right\} \\
&= \int Dt \int D\omega \int_0^\infty dw_1 \int \frac{dx_1}{2\pi} \int Dy_1 \exp\left\{-ix_1\left(\sqrt{1-q}y_1 + \sqrt{q}t - w_1\right)\right\} \\
&\times \prod_\gamma \int_0^\infty dw'_\gamma \int \frac{dx'_\gamma}{2\pi} \exp\left\{-\sum_\gamma\left[\frac{1-r}{2}x'^2_\gamma + ix'_\gamma\left(\frac{p-p'}{\sqrt{1-q}}y_1 + \sqrt{v_\omega}\omega + \frac{p'}{\sqrt{q}}t - w'_\gamma\right)\right]\right\} \\
&\times \prod_{a>1} \int_0^\infty dw_a \int \frac{dx_a}{2\pi} \exp\left\{-\sum_a\left[\frac{1-q}{2}x_a^2 + ix_a\left(\sqrt{q}t - w_a\right)\right]\right\} \\
&= \int Dt \int D\omega \int_0^\infty dw_1 \int \frac{dx_1}{2\pi} \int Dy_1 \exp\left\{-ix_1\left(\sqrt{1-q}y_1 + \sqrt{q}t - w_1\right)\right\} \\
&\times \prod_\gamma \int_0^\infty dw'_\gamma \frac{1}{\sqrt{2\pi(1-r)}} \exp\left\{-\sum_\gamma\left[\frac{\left(-h(\omega,t,y_1)\sqrt{1-r} - w'_\gamma\right)^2}{2(1-r)}\right]\right\} \\
&\times \prod_{a>1} \int_0^\infty dw_a \frac{1}{\sqrt{2\pi(1-q)}} \exp\left\{-\sum_a \frac{\left(\sqrt{q}t - w_a\right)^2}{2(1-q)}\right\} \\
&= \int Dt \int D\omega \int Dy_1 \prod_{a>1} \int_0^\infty dw_a \frac{1}{\sqrt{2\pi(1-q)}} \exp\left\{-\sum_a \frac{\left(\sqrt{q}t - w_a\right)^2}{2(1-q)}\right\} \prod_\gamma \int_0^\infty dw'_\gamma \frac{1}{\sqrt{2\pi(1-r)}} \exp\left\{-\sum_\gamma\left[\frac{\left(-h(\omega,t,y_1)\sqrt{1-r} - w'_\gamma\right)^2}{2(1-r)}\right]\right\} \\
&\times \int_0^\infty dw_1 \int \frac{dx_1}{2\pi} \exp\left\{-ix_1\left(\sqrt{1-q}y_1 + \sqrt{q}t - w_1\right)\right\} \\
&= \int Dt \int D\omega \int Dy_1 \prod_{a>1} \int_{\tilde{t}}^\infty D\tilde{w}_a \prod_\gamma \int_{h(\omega,t,y_1)}^\infty d\tilde{w}'_\gamma \Theta\left(\sqrt{1-q}y_1 + \sqrt{q}t\right) \\
&= \int Dt \int D\omega \int_{\tilde{t}}^\infty Dy H^{n-1}\left(\tilde{t}\right) H^m\left(h(\omega,t,y)\right) \qquad (14)
\end{aligned}
$$

The first term of the free energy [Eq. (8)] is then,

$$\left\langle \prod_{\mu}\left[\prod_{a,\gamma}\Theta\left(u_a^{\mu}\right)\Theta\left(v_{\gamma}^{\mu}\right)\right]\right\rangle = \left[\int Dt\int D\omega\int_{\tilde{t}}^{\infty} DyH^{n-1}\left(\tilde{t}\right)H^{m}\left(h\left(\omega,t,y_1\right)\right)\right]^{\alpha N}. \qquad (15)$$

The third term of Eq. (8) involves the weights explicitly. We compute it by defining the integral $\mathcal{I}_2$ below and using a similar strategy to HK14, modified to have Gaussian weight distributions instead of binary ones:

$$\begin{aligned}
\mathcal{I}_2^N &= \int DJ_i^a\int Dw_i^{\gamma}\exp\left(\sum_{a<b}\hat{Q}_{ab}\sum_i J_i^aJ_i^b+\sum_{a,\gamma}\hat{P}_{a\gamma}\sum_i J_i^aw_i^{\gamma}+\sum_{\gamma<\eta}\hat{R}_{\gamma\eta}\sum_i w_i^{\gamma}w_i^{\eta}\right)\\
&= \left[\int DJ^a\int Dw^{\gamma}\exp\left(\hat{q}\sum_{a<b}J^aJ^b+\left(\hat{p}-\hat{p}'\right)J^1\sum_{\gamma}w^{\gamma}+\hat{p}'\sum_{a,\gamma}J^aw^{\gamma}+\hat{r}\sum_{\gamma<\eta}w^{\gamma}w^{\eta}\right)\right]^N\\
\mathcal{I}_2 &= \int DJ^a\int Dw^{\gamma}\exp\left(\hat{q}\sum_{a<b}J^aJ^b+\left(\hat{p}-\hat{p}'\right)J^1\sum_{\gamma}w^{\gamma}+\hat{p}'\sum_{a,\gamma}J^aw^{\gamma}+\hat{r}\sum_{\gamma<\eta}w^{\gamma}w^{\eta}\right)\\
&= \int DJ^a\int Dw^{\gamma}\exp\left\{\frac{\hat{q}}{2}\left[\left(\sum_a J^a\right)^2-\sum_a\left(J^a\right)^2\right]+\left(\hat{p}-\hat{p}'\right)J^1\sum_{\gamma}w^{\gamma}+\frac{\hat{p}'}{2}\left[\left(\sum_a J^a+\sum_{\gamma}w^{\gamma}\right)^2-\left(\sum_a J^a\right)^2-\left(\sum_{\gamma}w^{\gamma}\right)^2\right]+\frac{\hat{r}}{2}\left[\left(\sum_{\gamma}w^{\gamma}\right)^2-\sum_{\gamma}\left(w^{\gamma}\right)^2\right]\right\}\\
&= \int DJ^a\int Dw^{\gamma}\exp\left\{\frac{\hat{q}-\hat{p}'}{2}\left(\sum_a J^a\right)^2-\frac{\hat{q}}{2}\sum_a\left(J^a\right)^2+\left(\hat{p}-\hat{p}'\right)J^1\sum_{\gamma}w^{\gamma}+\frac{\hat{p}'}{2}\left(\sum_a J^a+\sum_{\gamma}w^{\gamma}\right)^2+\frac{\hat{r}-\hat{p}'}{2}\left(\sum_{\gamma}w^{\gamma}\right)^2-\frac{\hat{r}}{2}\sum_{\gamma}\left(w^{\gamma}\right)^2\right\}\\
&= \int Dz_1\int Dz_3\int DJ^a\int Dw^{\gamma}\exp\left\{z_1\sqrt{\hat{q}-\hat{p}'}\sum_a J^a-\frac{\hat{q}}{2}\sum_a\left(J^a\right)^2+\left(\hat{p}-\hat{p}'\right)J^1\sum_{\gamma}w^{\gamma}+z_3\sqrt{\hat{p}'}\left(\sum_a J^a+\sum_{\gamma}w^{\gamma}\right)+\frac{\hat{r}-\hat{p}'}{2}\left(\sum_{\gamma}w^{\gamma}\right)^2-\frac{\hat{r}}{2}\sum_{\gamma}\left(w^{\gamma}\right)^2\right\}\\
&= \int Dz_1\int Dz_3\int Dw^{\gamma}\int DJ^1\exp\left\{\left(z_1\sqrt{\hat{q}-\hat{p}'}+z_3\sqrt{\hat{p}'}\right)J^1-\frac{\hat{q}}{2}\left(J^1\right)^2+\left[\left(\hat{p}-\hat{p}'\right)J^1+z_3\sqrt{\hat{p}'}\right]\sum_{\gamma}w^{\gamma}+\frac{\hat{r}-\hat{p}'}{2}\left(\sum_{\gamma}w^{\gamma}\right)^2-\frac{\hat{r}}{2}\sum_{\gamma}\left(w^{\gamma}\right)^2\right\}\\
&\quad\times\int DJ^{a>1}\exp\left[\left(z_1\sqrt{\hat{q}-\hat{p}'}+z_3\sqrt{\hat{p}'}\right)\sum_{a>1}J^a-\frac{\hat{q}}{2}\sum_{a>1}\left(J^a\right)^2\right]\\
&= \int Dz_1\int Dz_3\int Dw^{\gamma}\int DJ^1\exp\left\{\hat{a}J^1-\frac{\hat{q}}{2}\left(J^1\right)^2+\left[\left(\hat{p}-\hat{p}'\right)J^1+z_3\sqrt{\hat{p}'}\right]\sum_{\gamma}w^{\gamma}+\frac{\hat{r}-\hat{p}'}{2}\left(\sum_{\gamma}w^{\gamma}\right)^2-\frac{\hat{r}}{2}\sum_{\gamma}\left(w^{\gamma}\right)^2\right\}\\
&\quad\times\int\frac{dJ^{a>1}}{\sqrt{2\pi}}\exp\left[\sum_{a>1}\left(-\frac{1+\hat{q}}{2}\sum_{a>1}\left(J^a\right)^2+\hat{a}J^a\right)\right]\\
&= \int Dz_1\int Dz_3\int Dw^{\gamma}\exp\left\{z_3\sqrt{\hat{p}'}\sum_{\gamma}w^{\gamma}+\frac{\hat{r}-\hat{p}'}{2}\left(\sum_{\gamma}w^{\gamma}\right)^2-\frac{\hat{r}}{2}\sum_{\gamma}\left(w^{\gamma}\right)^2\right\}\int\frac{dJ}{\sqrt{2\pi}}\exp\left\{-\frac{1+\hat{q}}{2}J^2+\left(\hat{a}+\left(\hat{p}-\hat{p}'\right)\sum_{\gamma}w^{\gamma}\right)J\right\}\\
&\quad\times\left[\frac{1}{\sqrt{1+\hat{q}}}\exp\left(\frac{\hat{a}^2}{2\left(1+\hat{q}\right)}\right)\right]^{n-1}\\
&= \frac{1}{\sqrt{1+\hat{q}}^{n}}\int Dz_1\int Dz_3\left[\exp\left(\frac{\hat{a}^2}{2\left(1+\hat{q}\right)}\right)\right]^{n-1}\int Dw^{\gamma}\exp\left\{z_3\sqrt{\hat{p}'}\sum_{\gamma}w^{\gamma}+\frac{\hat{r}-\hat{p}'}{2}\left(\sum_{\gamma}w^{\gamma}\right)^2-\frac{\hat{r}}{2}\sum_{\gamma}\left(w^{\gamma}\right)^2+\frac{\left(\hat{a}+\left(\hat{p}-\hat{p}'\right)\sum_{\gamma}w^{\gamma}\right)^2}{2\left(1+\hat{q}\right)}\right\}
\end{aligned}$$

$$= \frac{1}{\sqrt{1+\hat{q}}^{n}} \int Dz_1 \int Dz_3 \left[\exp\left(\frac{\hat{a}^2}{2(1+\hat{q})}\right)\right]^n \int Dw^\gamma \exp\left\{\left[z_3\sqrt{\hat{p}'} + \frac{\hat{a}(\hat{p}-\hat{p}')}{1+\hat{q}}\right]\sum_\gamma w^\gamma + \frac{1}{2}\left[\hat{r}-\hat{p}'+\frac{(\hat{p}-\hat{p}')^2}{(1+\hat{q})}\right]\left(\sum_\gamma w^\gamma\right)^2 - \frac{\hat{r}}{2}\sum_\gamma (w^\gamma)^2\right\}$$

$$= \frac{1}{\sqrt{1+\hat{q}}^{n}} \int Dz_1 \int Dz_2 \int Dz_3 \left[\exp\left(\frac{\hat{a}^2}{2(1+\hat{q})}\right)\right]^n \int Dw^\gamma \exp\left\{\left[z_3\sqrt{\hat{p}'} + z_2\sqrt{\hat{r}-\hat{p}'+\frac{(\hat{p}-\hat{p}')^2}{(1+\hat{q})}} + \frac{\hat{a}(\hat{p}-\hat{p}')}{1+\hat{q}}\right]\sum_\gamma w^\gamma - \frac{\hat{r}}{2}\sum_\gamma (w^\gamma)^2\right\}$$

$$= \frac{1}{\sqrt{1+\hat{q}}^{n}} \int Dz_1 \int Dz_2 \int Dz_3 \left[\exp\left(\frac{\hat{a}^2}{2(1+\hat{q})}\right)\right]^n \left[\int \frac{dw}{\sqrt{2\pi}} \exp\left[\left(\hat{a}' + \frac{\hat{a}(\hat{p}-\hat{p}')}{1+\hat{q}}\right)w - \frac{1+\hat{r}}{2}w^2\right]\right]^m$$

$$= \frac{1}{\sqrt{1+\hat{q}}^{n}\sqrt{1+\hat{r}}^{m}} \int Dz_1 \int Dz_2 \int Dz_3 \exp\left(\frac{n\hat{a}^2}{2(1+\hat{q})}\right) \exp\left(\frac{m\left(\hat{a}' + \frac{\hat{a}(\hat{p}-\hat{p}')}{1+\hat{q}}\right)^2}{2(1+\hat{r})}\right) \tag{16}$$

Where we have defined

$$\begin{aligned} \hat{a} &= z_1\sqrt{\hat{q}-\hat{p}'} + z_3\sqrt{\hat{p}'} \\ \hat{a}' &= z_3\sqrt{\hat{p}'} + z_2\sqrt{\hat{r}-\hat{p}'+\frac{(\hat{p}-\hat{p}')^2}{1+\hat{q}}}. \end{aligned} \tag{17}$$

Note that the definition of $\hat{a}'$ is different here compared to [HK14].

Collecting terms of Eq. (8), taking the derivative with respect to m and taking the limits $n, m \to 0$, $N \to \infty$ the free energy reads,

$$\begin{aligned} F(x)/N = & \lim_{\substack{n\to 0\\ m\to 0}} \frac{\partial}{\partial m} \\ & \left[\alpha \log\left[\int Dt \int D\omega \int_{\tilde{t}}^{\infty} Dy H^{n-1}\left(\tilde{t}\right) H^m\left(h(\omega,t,y)\right)\right]\right. \\ & - \left[\frac{1}{2}n(n-1)q\hat{q} + mp\hat{p} + (n-1)mp'\hat{p}' + \frac{1}{2}m(m-1)r\hat{r} - xmp + \frac{n}{2}\log(1+\hat{q}) + \frac{m}{2}\log(1+\hat{r})\right] \\ & \left. + \log\left[\int D\mathbf{z} \exp\left(\frac{n\hat{a}^2}{2(1+\hat{q})}\right)\exp\left(\frac{m\left(\hat{a}'+\hat{a}(\hat{p}-\hat{p}')/(1+\hat{q})\right)^2}{2(1+\hat{r})}\right)\right]\right] \\ = & \lim_{\substack{n\to 0\\ m\to 0}} \left[\alpha \frac{\int Dt \int D\omega \int_{\tilde{t}}^{\infty} Dy H^{n-1}\left(\tilde{t}\right) H^m\left(h(\omega,t,y)\right)\log H\left(h(\omega,t,y)\right)}{\int Dt \int D\omega \int_{\tilde{t}}^{\infty} Dy H^{n-1}\left(\tilde{t}\right) H^m\left(h(\omega,t,y)\right)}\right. \\ & - \left[p\hat{p} + (n-1)p'\hat{p}' + \frac{1}{2}(2m-1)r\hat{r} - xp + \frac{1}{2}\log(1+\hat{r})\right] \\ & \left. + \frac{\int D\mathbf{z}\exp\left(\frac{m\left(\hat{a}'+\hat{a}\left(\hat{p}-\hat{p}'\right)/(1+\hat{q})\right)^2}{2(1+\hat{r})}\right)\frac{\left(\hat{a}'+\hat{a}\left(\hat{p}-\hat{p}'\right)/(1+\hat{q})\right)^2}{2(1+\hat{r})}}{\int D\mathbf{z}\exp\left(\frac{m(\hat{a}'+\hat{a}(\hat{p}-\hat{p}')/(1+\hat{q}))^2}{2(1+\hat{r})}\right)}\right] \end{aligned} \tag{18}$$

The normalized free energy is

$$f(x) = \left[\alpha \int Dt H^{-1}\left(\tilde{t}\right) \int D\omega \int_{\tilde{t}}^{\infty} Dy \log H\left(h(\omega,t,y)\right) - p\hat{p} + p'\hat{p}' + \frac{r\hat{r}}{2} + xp - \frac{1}{2}\log(1+\hat{r})\right.$$

$$+\frac{1}{2\left(1+\hat{r}\right)}\int D\mathbf{z}\left(\hat{a}'+\hat{a}\left(\hat{p}-\hat{p}'\right)/\left(1+\hat{q}\right)\right)^{2}\Bigg] \tag{19}$$

The continuity of the distributions of J, w implies that the integral $\int D\mathbf{z}\ldots$ can be carried out,

$$\begin{aligned}
\int D\mathbf{z}\left(\hat{a}'+\hat{a}\frac{\hat{p}-\hat{p}'}{1+\hat{q}}\right)^{2} &= \int Dz_1\int Dz_2\int Dz_3\left(\hat{a}'^2+\hat{a}^2\frac{\left(\hat{p}-\hat{p}'\right)^2}{\left(1+\hat{q}\right)^2}+2\frac{\hat{p}-\hat{p}'}{1+\hat{q}}\hat{a}\hat{a}'\right)\\
&= \int Dz_1\int Dz_2\int Dz_3\left[\left(z_3\sqrt{\hat{p}'}+z_2\sqrt{\hat{r}-\hat{p}'+\frac{\left(\hat{p}-\hat{p}'\right)^2}{1+\hat{q}}}\right)^2+\left(z_1\sqrt{\hat{q}-\hat{p}'}+z_3\sqrt{\hat{p}'}\right)^2\frac{\left(\hat{p}-\hat{p}'\right)^2}{\left(1+\hat{q}\right)^2}\right.\\
&\quad\left.+2\frac{\hat{p}-\hat{p}'}{1+\hat{q}}\left(z_1\sqrt{\hat{q}-\hat{p}'}+z_3\sqrt{\hat{p}'}\right)\left(z_3\sqrt{\hat{p}'}+z_2\sqrt{\hat{r}-\hat{p}'+\frac{\left(\hat{p}-\hat{p}'\right)^2}{1+\hat{q}}}\right)\right]\\
&= \hat{p}'+\hat{r}-\hat{p}'+\frac{\left(\hat{p}-\hat{p}'\right)^2}{1+\hat{q}}+\left(\hat{q}-\hat{p}'+\hat{p}'\right)\frac{\left(\hat{p}-\hat{p}'\right)^2}{\left(1+\hat{q}\right)^2}+2\hat{p}'\frac{\hat{p}-\hat{p}'}{1+\hat{q}}\\
&= \hat{r}+\frac{\left(\hat{p}-\hat{p}'\right)^2}{1+\hat{q}}+\hat{q}\frac{\left(\hat{p}-\hat{p}'\right)^2}{\left(1+\hat{q}\right)^2}+2\hat{p}'\frac{\hat{p}-\hat{p}'}{1+\hat{q}}\\
&= \hat{r}+\hat{q}\frac{\left(\hat{p}-\hat{p}'\right)^2}{\left(1+\hat{q}\right)^2}+\frac{\hat{p}^2-\hat{p}'^2}{1+\hat{q}}\\
&= \hat{r}+\frac{\left(\hat{p}-\hat{p}'\right)\left(\hat{p}+\hat{p}'+2\hat{q}\hat{p}\right)}{\left(1+\hat{q}\right)^2} \qquad (20)
\end{aligned}$$

So the free energy is finally (analogous to Eq. A8 in [HK14]),

$$\begin{aligned}
f\left(x\right) &= \alpha\int DtH^{-1}\left(\tilde{t}\right)\int D\omega\int_{\tilde{t}}^{\infty}Dy\log H\left(h\left(\omega,t,y\right)\right)-p\hat{p}+p'\hat{p}'+\frac{r\hat{r}}{2}+xp-\frac{1}{2}\log\left(1+\hat{r}\right)\\
&\quad+\frac{1}{2\left(1+\hat{r}\right)}\left[\hat{r}+\frac{\left(\hat{p}-\hat{p}'\right)\left(\hat{p}+\hat{p}'+2\hat{q}\hat{p}\right)}{\left(1+\hat{q}\right)^2}\right] \qquad (21)
\end{aligned}$$

The derivatives of $f\left(x\right)$ w.r.t. the order parameters and the conjugate variables give the saddle point equations. These give an 8 dimensional system of equations for the order parameters, which must be solved to understand if solutions exist.

The following derivatives are useful in computing the saddle point equations (Eq. A9a,...,h in [HK14]).

$$\begin{aligned}
\partial_q\tilde{t} &= -t\partial_q\frac{\sqrt{q}}{\sqrt{1-q}}\\
&= -t\frac{\frac{\sqrt{1-q}}{2\sqrt{q}}+\frac{\sqrt{q}}{2\sqrt{1-q}}}{1-q}\\
&= -\frac{t}{2\sqrt{q}\left(1-q\right)^{\frac{3}{2}}}\\
\frac{d}{dt}h\left(\omega,t,y\right) &= -\frac{p'}{\sqrt{q\left(1-r\right)}}\\
\frac{d}{dy}h\left(\omega,t,y\right) &= -\frac{p-p'}{\sqrt{\left(1-q\right)\left(1-r\right)}}
\end{aligned}$$

$$
\begin{aligned}
\frac{d}{d\omega}h\left(\omega,t,y\right) &= -\frac{1}{\sqrt{1-r}}\sqrt{r-\frac{p'^2}{q}-\frac{\left(p-p'\right)^2}{1-q}} \\
&= -\frac{\sqrt{v_\omega}}{\sqrt{1-r}} \\
\partial_q h\left(\omega,t,y\right) &= -\frac{1}{\sqrt{1-r}}\partial_q\left[\frac{p-p'}{\sqrt{1-q}}y+\sqrt{r-\frac{p'^2}{q}-\frac{\left(p-p'\right)^2}{1-q}}\omega+\frac{p'}{\sqrt{q}}t\right] \\
&= -\frac{1}{2\sqrt{1-r}}\left[-\frac{p-p'}{\left(1-q\right)^{\frac{3}{2}}}y+\frac{\frac{p'^2}{q^2}-\frac{\left(p-p'\right)^2}{\left(1-q\right)^2}}{\sqrt{v_\omega}}\omega-\frac{p'}{q^{\frac{3}{2}}}t\right] \\
\partial_p h\left(\omega,t,y\right) &= -\frac{1}{\sqrt{1-r}}\partial_p\left[\frac{p-p'}{\sqrt{1-q}}y+\sqrt{r-\frac{p'^2}{q}-\frac{\left(p-p'\right)^2}{1-q}}\omega+\frac{p'}{\sqrt{q}}t\right] \\
&= -\frac{1}{\sqrt{1-r}}\left[\frac{1}{\sqrt{1-q}}y-\frac{\frac{p-p'}{1-q}}{\sqrt{v_\omega}}\omega\right] \\
\partial_{p'} h\left(\omega,t,y\right) &= -\frac{1}{\sqrt{1-r}}\partial_{p'}\left[\frac{p-p'}{\sqrt{1-q}}y+\sqrt{r-\frac{p'^2}{q}-\frac{\left(p-p'\right)^2}{1-q}}\omega+\frac{p'}{\sqrt{q}}t\right] \\
&= -\frac{1}{\sqrt{1-r}}\left[-\frac{1}{\sqrt{1-q}}y-\frac{\frac{p'}{q}-\frac{p-p'}{1-q}}{\sqrt{v_\omega}}\omega+\frac{1}{\sqrt{q}}t\right] \\
\partial_r h\left(\omega,t,y\right) &= \frac{1}{2\left(1-r\right)^{\frac{3}{2}}}\left[\frac{p-p'}{\sqrt{1-q}}y+\sqrt{v_\omega}\omega+\frac{p'}{\sqrt{q}}t\right]-\frac{1}{2\sqrt{1-r}\sqrt{v_\omega}}\omega \\
&= -\frac{h\left(\omega,t,y\right)}{2\left(1-r\right)}-\frac{\omega}{2\sqrt{1-r}\sqrt{v_\omega}}
\end{aligned} \tag{22}
$$

Additionally we define $\mathcal{R}\left(x\right)=\frac{G(x)}{H(x)}$ and compute the following integrals,

$$
\begin{aligned}
\int_{\tilde{t}}^{\infty} Dyy\mathcal{R}\left(h\right) &= -\int_{\tilde{t}}^{\infty} dy\frac{dG\left(y\right)}{dy}\mathcal{R}\left(h\right) \\
&= -\int_{\tilde{t}}^{\infty} dy\left\{\frac{d}{dy}\left[G\left(y\right)\mathcal{R}\left(h\right)\right]-G\left(y\right)\frac{d\mathcal{R}\left(h\right)}{dh}\frac{dh}{dy}\right\} \\
&= G\left(\tilde{t}\right)\mathcal{R}\left(h\left(\omega,t,y=\tilde{t}\right)\right)-\frac{p-p'}{\sqrt{\left(1-q\right)\left(1-r\right)}}\int_{\tilde{t}}^{\infty} Dy\mathcal{R}'\left(h\right) \\
\int D\omega\omega\mathcal{R}\left(h\right) &= -\frac{\sqrt{v_\omega}}{\sqrt{1-r}}\int D\omega\mathcal{R}'\left(h\right) \\
\int DttH^{-1}\left(\tilde{t}\right)\int_{\tilde{t}}^{\infty} Dy\mathcal{R}\left(h\right) &= \int Dt\frac{d}{dt}\left[H^{-1}\left(\tilde{t}\right)\int_{\tilde{t}}^{\infty} Dy\mathcal{R}\left(h\right)\right] \\
&= +\frac{\sqrt{q}}{\sqrt{1-q}}\int DtH^{-2}\left(\tilde{t}\right)G\left(\tilde{t}\right)\int_{\tilde{t}}^{\infty} Dy\mathcal{R}\left(h\right) \\
&\quad +\frac{\sqrt{q}}{\sqrt{1-q}}\int DtH^{-1}\left(\tilde{t}\right)G\left(\tilde{t}\right)\mathcal{R}\left(h\left(\omega,t,y=\tilde{t}\right)\right)
\end{aligned}
$$

$$-\frac{p'}{\sqrt{q(1-r)}}\int DtH^{-1}(\tilde{t})\int_{\tilde{t}}^{\infty}Dy\mathcal{R}'(h) \tag{23}$$

The saddle equations $\partial_q f = 0$, $\partial_{\hat{q}} f = 0$ cannot depend on the new order parameters introduced in the calculation of the Franz-Parisi energy in [HK14]. This order parameter (and its conjugate) describes the original solution J, which does not care about the overlap with the weight vector w. Eqs. (A9a, A9b) in [HK14] can be obtained by taking the derivatives of the free energy without the additional order parameters (r, p, p') and their conjugates. For the binary synapses, this free energy reads (Eq. 11 in Krauth, Mezard 1989, J Phys France, taking the limit $\beta \to \infty$, and setting $\kappa = 0$),

$$f = \frac{1}{2}\hat{q}(q-1) + \int Dz \log\left[2\cosh\left(z\sqrt{\hat{q}}\right)\right] + \alpha\int Dt \log H\left(\frac{\sqrt{q}}{\sqrt{1-q}}t\right) \tag{24}$$

For the continuous case [G88] with $\kappa = 0$ (for an easier references, see Eqs. 10.107 and 10.109 in Introduction to the theory of neural computation by Hertz, Krogh, Palmer, where $E \to \hat{q}$)

$$\begin{aligned}
\hat{q} &= \frac{q}{(1-q)^2} \\
\frac{q^{\frac{3}{2}}}{\sqrt{1-q}} &= \alpha\int Dt t\mathcal{R}(\tilde{t}) \\
&= \alpha\sqrt{q(1-q)}\int Dt\mathcal{R}^2\left(\frac{\sqrt{q}}{\sqrt{1-q}}t\right) \\
\frac{q^2}{1-q} &= \alpha\int Dt\mathcal{R}^2\left(\frac{\sqrt{q}}{\sqrt{1-q}}t\right)
\end{aligned} \tag{25}$$

3.1 Computation of $\partial_q f$ (binary case)

$$\begin{aligned}
\partial_q f &= 0 \\
&= \frac{1}{2}\hat{q} - \frac{\alpha}{2\sqrt{q}(1-q)^{\frac{3}{2}}}\int Dt t\mathcal{R}\left(\frac{\sqrt{q}}{\sqrt{1-q}}t\right) \\
\int Dt t\mathcal{R}(kt) &= \int Dt\frac{d}{dt}\mathcal{R}(kt) \\
&= k\int Dt\frac{-ktG(kt)H(kt)+G^2(kt)}{H^2(kt)} \\
&= k\int Dt\left[\mathcal{R}^2(kt) - kt\mathcal{R}(kt)\right] \\
&= k\int Dt\mathcal{R}^2(kt) - k^2\int Dt t\mathcal{R}(kt) \\
\int Dt t\mathcal{R}(kt) &= \frac{k}{1+k^2}\int Dt\mathcal{R}^2(kt) \\
&= \frac{\frac{\sqrt{q}}{\sqrt{1-q}}}{1+\frac{q}{1-q}}\int Dt\mathcal{R}^2\left(\frac{\sqrt{q}}{\sqrt{1-q}}t\right) \\
&= \sqrt{q(1-q)}\int Dt\mathcal{R}^2\left(\frac{\sqrt{q}}{\sqrt{1-q}}t\right) \\
\hat{q} &= \frac{\alpha}{1-q}\int Dt\mathcal{R}^2\left(\frac{\sqrt{q}}{\sqrt{1-q}}t\right)
\end{aligned} \tag{26}$$

3.2 Computation of $\partial_{\hat{q}} f$ (binary case)

$$
\begin{aligned}
\partial_{\hat{q}} f &= 0 \\
&= \frac{1}{2}(q-1) + \frac{1}{2\sqrt{\hat{q}}} \int Dz z \tanh\left(z\sqrt{\hat{q}}\right) \\
&= \frac{1}{2}(q-1) + \frac{1}{2\sqrt{\hat{q}}} \int Dz \frac{d}{dz} \tanh\left(z\sqrt{\hat{q}}\right) \\
&= \frac{1}{2}\left(q + \int Dz \frac{1-\cosh^2\left(z\sqrt{\hat{q}}\right)}{\cosh^2\left(z\sqrt{\hat{q}}\right)}\right) \\
&= \frac{1}{2}\left(q - \int Dz \tanh^2\left(z\sqrt{\hat{q}}\right)\right) \\
q &= \int Dz \tanh^2\left(z\sqrt{\hat{q}}\right)
\end{aligned} \tag{27}
$$

3.3 Computation of $\partial_p f$

$$
\begin{aligned}
\partial_p f(x) &= 0 \\
&= \alpha \int Dt H^{-1}\left(\tilde{t}\right) \int D\omega \int_{\tilde{t}}^{\infty} Dy \partial_p \log H\left(h\left(\omega, t, y\right)\right) - \hat{p} + x \\
\hat{p} - x &= \alpha \int Dt H^{-1}\left(\tilde{t}\right) \int D\omega \int_{\tilde{t}}^{\infty} Dy \left(\partial_p h\right) \frac{d}{dh} \log H(h) \\
&= \frac{\alpha}{\sqrt{1-r}} \int Dt H^{-1}\left(\tilde{t}\right) \int D\omega \int_{\tilde{t}}^{\infty} Dy \left[\frac{1}{\sqrt{1-q}} y - \frac{p-p'}{(1-q)\sqrt{v_\omega}} \omega\right] \mathcal{R}(h) \\
&= \frac{\alpha}{\sqrt{(1-q)(1-r)}} \int Dt H^{-1}\left(\tilde{t}\right) \int D\omega \int_{\tilde{t}}^{\infty} Dy y \mathcal{R}(h) \\
&\quad - \frac{\alpha(p-p')}{\sqrt{v_\omega}(1-q)\sqrt{1-r}} \int Dt H^{-1}\left(\tilde{t}\right) \int_{\tilde{t}}^{\infty} Dy \int D\omega \omega \mathcal{R}(h) \\
&= \frac{\alpha}{\sqrt{(1-q)(1-r)}} \int D\omega \int Dt H^{-1}\left(\tilde{t}\right) \left[G\left(\tilde{t}\right) \mathcal{R}\left(h\left(\omega, t, y = \tilde{t}\right)\right) - \frac{p-p'}{\sqrt{(1-q)(1-r)}} \int_{\tilde{t}}^{\infty} Dy \mathcal{R}'(h)\right] \\
&\quad + \frac{\alpha(p-p')}{(1-q)(1-r)} \int Dt H^{-1}\left(\tilde{t}\right) \int_{\tilde{t}}^{\infty} Dy \int D\omega \mathcal{R}'(h) \\
&= \frac{\alpha}{\sqrt{(1-q)(1-r)}} \int D\omega \int Dt \mathcal{R}\left(\tilde{t}\right) \mathcal{R}\left(h\left(\omega, t, y = \tilde{t}\right)\right)
\end{aligned} \tag{28}
$$

3.4 Computation of $\partial_r f$

$$
\begin{aligned}
\partial_r f(x) &= 0 \\
&= \frac{\hat{r}}{2} + \alpha \int Dt H^{-1}\left(\tilde{t}\right) \int D\omega \int_{\tilde{t}}^{\infty} Dy \partial_r \log H\left(h\left(\omega, t, y\right)\right) \\
\partial_r \log H\left(h\left(\omega, t, y\right)\right) &= \frac{G(h)}{H(h)} \partial_r h
\end{aligned}
$$

$$
\begin{aligned}
&= -\mathcal{R}(h)\left[\frac{h}{2(1-r)} - \frac{\omega}{2\sqrt{1-r}\sqrt{v_\omega}}\right] \\
\int DtH^{-1}(\tilde{t})\int D\omega\int_{\tilde{t}}^{\infty} Dy\partial_r \log H(h) &= \int DtH^{-1}(\tilde{t})\int D\omega\int_{\tilde{t}}^{\infty} Dy\mathcal{R}(h)\left[\frac{h}{2(1-r)} - \frac{\omega}{2\sqrt{1-r}\sqrt{v_\omega}}\right] \\
&= \int DtH^{-1}(\tilde{t})\int D\omega\int_{\tilde{t}}^{\infty} Dy\left[\frac{h\mathcal{R}(h)}{2(1-r)} - \frac{\frac{d}{d\omega}\mathcal{R}(h)}{2\sqrt{1-r}\sqrt{v_\omega}}\right] \\
&= \int DtH^{-1}(\tilde{t})\int D\omega\int_{\tilde{t}}^{\infty} Dy\left[\frac{h\mathcal{R}(h)}{2(1-r)} - \frac{\mathcal{R}'(h)\frac{dh}{d\omega}}{2\sqrt{1-r}\sqrt{v_\omega}}\right] \\
&= \int DtH^{-1}(\tilde{t})\int D\omega\int_{\tilde{t}}^{\infty} Dy\left[\frac{h\mathcal{R}(h)}{2(1-r)} + \frac{\sqrt{v_\omega}}{\sqrt{1-r}}\frac{\mathcal{R}'(h)}{2\sqrt{1-r}\sqrt{v_\omega}}\right] \\
&= \frac{1}{2(1-r)}\int DtH^{-1}(\tilde{t})\int D\omega\int_{\tilde{t}}^{\infty} Dy\mathcal{R}^2(h) \\
\hat{r} &= \frac{1}{1-r}\int DtH^{-1}(\tilde{t})\int D\omega\int_{\tilde{t}}^{\infty} Dy\mathcal{R}^2(h) \qquad (29)
\end{aligned}
$$

3.5 Computation of $\partial_{p'} f$

$$
\begin{aligned}
\partial_{p'} f &= 0 \\
&= \hat{p}' + \alpha\int DtH^{-1}(\tilde{t})\int D\omega\int_{\tilde{t}}^{\infty} Dy\partial_{p'}\log H(h) \\
\int DtH^{-1}(\tilde{t})\int D\omega\int_{\tilde{t}}^{\infty} Dy\partial_{p'}\log H(h) &= \int DtH^{-1}(\tilde{t})\int D\omega\int_{\tilde{t}}^{\infty} Dy\mathcal{R}(h)\,\partial_{p'}h \\
&= -\frac{1}{\sqrt{1-r}}\int DtH^{-1}(\tilde{t})\int D\omega\int_{\tilde{t}}^{\infty} Dy\mathcal{R}(h)\left[-\frac{1}{\sqrt{1-q}}y - \frac{p'-pq}{q(1-q)\sqrt{v_\omega}}\omega + \frac{1}{\sqrt{q}}t\right] \\
&= \frac{1}{\sqrt{(1-q)(1-r)}}\int D\omega\int DtH^{-1}(\tilde{t})\left\{G(\tilde{t})\,\mathcal{R}\left(h\left(\omega,t,y=\tilde{t}\right)\right) - \frac{p-p'}{\sqrt{(1-q)(1-r)}}\int_{\tilde{t}}^{\infty} Dy\mathcal{R}'(h)\right\} \\
&\quad -\frac{p'-pq}{q(1-q)(1-r)}\int DtH^{-1}(\tilde{t})\int D\omega\int_{\tilde{t}}^{\infty} Dy\mathcal{R}'(h) \\
&\quad -\frac{1}{\sqrt{q(1-r)}}\int D\omega\int DtH^{-1}(\tilde{t})\left\{\frac{\sqrt{q}}{\sqrt{1-q}}\left[\mathcal{R}(\tilde{t})\int_{\tilde{t}}^{\infty} Dy\mathcal{R}(h) + G(\tilde{t})\,\mathcal{R}\left(h\left(\omega,t,y=\tilde{t}\right)\right)\right] - \frac{p'}{\sqrt{q(1-r)}}\int_{\tilde{t}}^{\infty} Dy\mathcal{R}'(h)\right\} \\
&= -\frac{p-p'}{(1-q)(1-r)}\int D\omega\int DtH^{-1}(\tilde{t})\int_{\tilde{t}}^{\infty} Dy\mathcal{R}'(h) \\
&\quad -\frac{p'/q-p}{(1-q)(1-r)}\int D\omega\int DtH^{-1}(\tilde{t})\int_{\tilde{t}}^{\infty} Dy\mathcal{R}'(h) \\
&\quad +\frac{p'/q-p'}{(1-q)(1-r)}\int D\omega\int DtH^{-1}(\tilde{t})\int_{\tilde{t}}^{\infty} Dy\mathcal{R}'(h) \\
&\quad -\frac{1}{\sqrt{(1-q)(1-r)}}\int D\omega\int DtH^{-1}(\tilde{t})\,\mathcal{R}(\tilde{t})\int_{\tilde{t}}^{\infty} Dy\mathcal{R}(h) \\
&= -\frac{1}{\sqrt{(1-q)(1-r)}}\int D\omega\int DtH^{-1}(\tilde{t})\,\mathcal{R}(\tilde{t})\int_{\tilde{t}}^{\infty} Dy\mathcal{R}(h)
\end{aligned}
$$

$$\hat{p}' = \frac{\alpha}{\sqrt{(1-q)(1-r)}} \int D\omega \int Dt H^{-1}\left(\tilde{t}\right) \mathcal{R}\left(\tilde{t}\right) \int_{\tilde{t}}^{\infty} Dy \mathcal{R}\left(h\right) \tag{30}$$

3.6 Computation of $\partial_{\hat{p}} f$

$$\begin{aligned}
\partial_{\hat{p}} f &= 0 \\
&= -p + \frac{\left(\hat{p} + \hat{p}' + 2\hat{q}\hat{p}\right) + \left(\hat{p} - \hat{p}'\right)\left(1 + 2\hat{q}\right)}{2\left(1+\hat{r}\right)\left(1+\hat{q}\right)^2} \\
p &= \frac{\hat{p} + \hat{p}' + 2\hat{q}\hat{p} + \hat{p} - \hat{p}' + 2\hat{q}\hat{p} - 2\hat{q}\hat{p}'}{2\left(1+\hat{r}\right)\left(1+\hat{q}\right)^2} \\
&= \frac{\hat{p}\left(1+2\hat{q}\right) - \hat{q}\hat{p}'}{\left(1+\hat{r}\right)\left(1+\hat{q}\right)^2}
\end{aligned} \tag{31}$$

3.7 Computation of $\partial_{\hat{p}'} f$

$$\begin{aligned}
\partial_{\hat{p}'} &= 0 \\
&= p' + \frac{-\hat{p} - \hat{p}' - 2\hat{q}\hat{p} + \hat{p} - \hat{p}'}{2\left(1+\hat{r}\right)\left(1+\hat{q}\right)^2} \\
p' &= \frac{\hat{p}' + \hat{q}\hat{p}}{\left(1+\hat{r}\right)\left(1+\hat{q}\right)^2}
\end{aligned} \tag{32}$$

3.8 Computation of $\partial_{\hat{r}} f$

$$\begin{aligned}
\partial_{\hat{r}} &= 0 \\
&= \frac{r}{2} - \frac{1}{2\left(1+\hat{r}\right)} + \frac{1}{2\left(1+\hat{r}\right)} - \frac{1}{2\left(1+\hat{r}\right)^2}\left[\hat{r} + \frac{\left(\hat{p}-\hat{p}'\right)\left(\hat{p}+\hat{p}'+2\hat{q}\hat{p}\right)}{\left(1+\hat{q}\right)^2}\right] \\
&= \frac{r}{2} - \frac{1}{2\left(1+\hat{r}\right)^2}\left[\hat{r} + \frac{\left(\hat{p}-\hat{p}'\right)\left(\hat{p}+\hat{p}'+2\hat{q}\hat{p}\right)}{\left(1+\hat{q}\right)^2}\right] \\
r &= \frac{1}{\left(1+\hat{r}\right)^2}\left[\hat{r} + \frac{\left(\hat{p}-\hat{p}'\right)\left(\hat{p}+\hat{p}'+2\hat{q}\hat{p}\right)}{\left(1+\hat{q}\right)^2}\right]
\end{aligned} \tag{33}$$

Collecting the saddle point equations (analogous to Eq. (A9a,...,h) in [HK14])

$$\begin{aligned}
\hat{q} &= \frac{q}{\left(1-q\right)^2} \\
\frac{q^2}{1-q} &= \alpha \int Dt \mathcal{R}^2\left(\frac{\sqrt{q}}{\sqrt{1-q}} t\right) \\
\hat{p} &= x + \frac{\alpha}{\sqrt{(1-q)(1-r)}} \int D\omega \int Dt \mathcal{R}\left(\tilde{t}\right) \mathcal{R}\left(h\left(\omega, t, y = \tilde{t}\right)\right) \\
p &= \frac{\hat{p}\left(1+2\hat{q}\right) - \hat{q}\hat{p}'}{\left(1+\hat{r}\right)\left(1+\hat{q}\right)^2}
\end{aligned}$$

$$
\begin{aligned}
\hat{p}' &= \frac{\alpha}{\sqrt{(1-q)(1-r)}} \int D\omega \int DtH^{-1}\left(\tilde{t}\right)\mathcal{R}\left(\tilde{t}\right)\int_{\tilde{t}}^{\infty} Dy\mathcal{R}\left(h\right) \\
p' &= \frac{\hat{p}' + \hat{q}\hat{p}}{\left(1+\hat{r}\right)\left(1+\hat{q}\right)^2} \\
\hat{r} &= \frac{1}{1-r}\int DtH^{-1}\left(\tilde{t}\right)\int D\omega \int_{\tilde{t}}^{\infty} Dy\mathcal{R}^2\left(h\right) \\
r &= 1 + \frac{1}{\left(1+\hat{r}\right)^2}\left[1 + 2\hat{r} + \frac{\left(\hat{p}-\hat{p}'\right)\left(\hat{p}+\hat{p}'+2\hat{q}\hat{p}\right)}{\left(1+\hat{q}\right)^2}\right]
\end{aligned} \tag{34}
$$